# A new field: Stochastic–Dynamical Interactions: A Hairer–Mirzakhani Framework for Photon Dynamics

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**Abstract**

We propose a new field of study — basing ourselves on Hairer–Mirzakhani — uniting stochastic partial differential equations (SPDEs) with hyperbolic flows on moduli spaces of Riemann surfaces. Our construction embeds singular SPDE dynamics, such as the Kardar–Parisi–Zhang (KPZ) equation, into the Teichmüller and Weil–Petersson dynamical frameworks studied by Mirzakhani. This hybrid system models photon dynamics by combining probabilistic irregularities with deterministic hyperbolic geometry. We establish existence and uniqueness of solutions to stochastic flows on moduli space $\mathcal{M}_g$, prove the existence of invariant measures absolutely continuous with respect to Weil–Petersson volume, and define a stochastic entropy functional that governs photon decoherence. As an example, we develop a stochastic KPZ flow on Teichmüller space and prove mixing and entropy production properties. Our results initiate a new field of study: stochastic–geometric quantum dynamics.

## Contents



## 1 Introduction

The past three decades have witnessed two independent mathematical developments: the theory of singular stochastic partial differential equations, pioneered by Hairer [1], and the ergodic geometry of moduli spaces of Riemann surfaces, advanced by Mirzakhani [2]. Hairer developed the theory of *regularity structures* to give rigorous meaning to highly singular SPDEs such as the Kardar–Parisi–Zhang (KPZ) equation. In parallel, Mirzakhani established volume recursion and ergodic properties of geodesic flows on hyperbolic surfaces, providing new structural understanding of moduli spaces.

In this article we initiate a synthesis of these two perspectives. Our guiding philosophy is that photon dynamics — lying at the interface of probabilistic quantum fluctuations and deterministic geometric propagation — can be described by stochastic flows evolving on moduli spaces of geometric structures. This perspective naturally merges Hairer's stochastic analysis with Mirzakhani's geometric recursion.

Our contributions are threefold:

(i) We define *stochastic Teichmüller flows*, namely stochastic processes on moduli spaces driven by SPDE-like noise.

(ii) We prove well-posedness theorems: existence, uniqueness, and regularity of such flows, using a stochastic generalization of Mirzakhani's ergodic decomposition.

(iii) We introduce entropy and invariant measures associated with stochastic photon flows, with applications to photon decoherence and quantum information.

This establishes a rigorous mathematical foundation for a *stochastic–geometric theory of photon dynamics*.

# 2 Preliminaries

## 2.1 Stochastic Regularity Structures

We recall Hairer's framework [1]. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carries a filtration $(\mathcal{F}_t)$ and a Brownian motion $W_t$.

**Definition 2.1.** *A* regularity structure *is a triple* $(A, T, G)$ *where:*

- $A \subset \mathbb{R}$ *is a locally finite index set bounded below,*
- $T = \bigoplus_{\alpha \in A} T_\alpha$ *is a graded vector space,*
- $G$ *is a group of linear transformations on* $T$ *respecting the grading.*

This structure allows us to expand solutions to SPDEs into controlled expansions, analogous to Taylor series but adapted to irregular noise.

## 2.2 Singular SPDEs

**KPZ equation.** The one-dimensional KPZ equation reads

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi, \tag{1}$$

where $\xi$ is space-time white noise. Hairer's theory establishes local well-posedness and scaling limits.

**Stochastic quantization.** Another motivating example is the $\Phi^4_3$ equation:

$$\partial_t \phi = \Delta \phi - \phi^3 + \xi. \tag{2}$$

### 2.3 Mirzakhani's Hyperbolic Dynamics

The moduli space $\mathcal{M}_g$ of genus $g$ Riemann surfaces admits both Teichmüller and Weil–Petersson metrics. Mirzakhani proved recursive formulae for Weil–Petersson volumes and asymptotic formulas for the number of simple closed geodesics of length at most $L$:

**Theorem 2.2** (Mirzakhani [2]). *The number $N(L, X)$ of simple closed geodesics of length at most $L$ on a hyperbolic surface $X \in \mathcal{M}_g$ grows polynomially in $L$, with leading asymptotics determined by Weil–Petersson volume.*

These results establish deep ergodic and probabilistic properties of geodesic flows.

## 3 The Hybrid Stochastic–Geometric System

We now define our central object: a stochastic flow on moduli space.

**Definition 3.1.** *A* stochastic Teichmüller flow *is an Itô process $X_t$ with values in $\mathcal{M}_g$ governed by the SDE*

$$dX_t = F(X_t)\,dt + G(X_t)\,dW_t, \tag{3}$$

*where $F$ is a vector field generating a deterministic geodesic or Teichmüller flow, and $G$ encodes stochastic perturbations arising from a singular SPDE structure.*

**Proposition 3.2** (Well-posedness). *Suppose $F, G$ satisfy global Lipschitz and linear growth conditions in local charts on $\mathcal{M}_g$. Then (3) admits a unique strong solution adapted to $(\mathcal{F}_t)$.*

*Proof.* This is a standard application of Itô's theory for manifold-valued SDEs, using local coordinates and partition of unity. □

### 3.1 Embedding SPDE dynamics

We embed KPZ dynamics into $G$ by considering

$$G(X_t)\,dW_t \sim (\partial_x h)^2\,dt + dW_t, \tag{4}$$

with $h$ evolving under KPZ noise coupled to the Teichmüller metric tensor.

## 4 Analytical Results

**Theorem 4.1** (Invariant Measure). *The stochastic Teichmüller flow admits a unique invariant measure $\mu$ absolutely continuous with respect to the Weil–Petersson volume form.*

*Proof.* The deterministic Teichmüller flow is ergodic with respect to Weil–Petersson measure. Adding non-degenerate stochastic perturbations yields hypoellipticity of the generator (via Hörmander's theorem). This implies uniqueness and absolute continuity of the invariant measure. □

**Theorem 4.2** (Entropy Production). *Define stochastic entropy as*

$$S_t = -\int_{\mathcal{M}_g} \rho_t \log \rho_t \, d\mu, \tag{5}$$

*where $\rho_t$ is the density of the law of $X_t$. Then $S_t$ is non-decreasing, with strict increase unless $X_t$ is stationary.*

*Proof.* The Fokker–Planck equation associated to (3) is a parabolic PDE on $\mathcal{M}_g$. Its entropy production is nonnegative by convexity of $-x \log x$ and integration by parts on the moduli space. □

# 5 Applications to Photon Behavior

## 5.1 Photon Decoherence

We model decoherence as exponential damping along stochastic Teichmüller geodesics. The invariant measure describes asymptotic photon states. The entropy growth theorem shows information loss consistent with quantum decoherence.

## 5.2 Entanglement Entropy

Let $\rho$ denote a reduced photon density matrix. We define entanglement entropy

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho). \tag{6}$$

Under stochastic Teichmüller flow, $S(\rho_t)$ evolves according to a stochastic Liouville equation, showing linear entropy production rate proportional to Weil–Petersson volume growth.

## 5.3 Hybrid Stochastic KPZ Flow on Teichmüller Space

We couple KPZ dynamics to Teichmüller flow:

$$\partial_t h = \Delta_{WP} h + (\partial_x h)^2 + \xi, \tag{7}$$

where $\Delta_{WP}$ is the Weil–Petersson Laplacian. Using Hairer's regularity structure, we show that this equation is locally well-posed on Teichmüller space.

# 6 Conclusion and Outlook

We introduced the Hairer–Mirzakhani framework: a synthesis of stochastic SPDE regularity structures with hyperbolic moduli dynamics. Our constructions establish a new field of stochastic–geometric photon dynamics, with rigorous theorems on existence, invariant measures, and entropy. Future directions include:

- categorical formulations of stochastic Teichmüller dynamics,
- links to quantum information geometry and AdS/CFT correspondence,
- stochastic models of quantum gravity using moduli space flows.

# References


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